ASCHAM SCHOOL

2019 YEAR 12
MATHEMATICS EXTENSION 2
TRIAL EXAM

GENERAL INSTRUCTIONS

Reading time – 5 minutes
Working time – 180 minutes
Use black pen, non-erasable
NESA-approved calculators may be used
Reference Sheet is provided

Total Marks - 40

Section A – Multiple Choice (1 mark each)

Attempt Questions 1 to 10.
Select answers on the separate multiple choice sheet provided.
Write your NESA number on the multiple choice sheet.

Section B – Questions 11 – 16 (15 marks each)

Start each question in a new booklet.
If you use a second booklet for a question, place it inside the first.
Label extra booklets for the same question as, for example, Q11-2 etc.
Write your NESA number and question number on each booklet.

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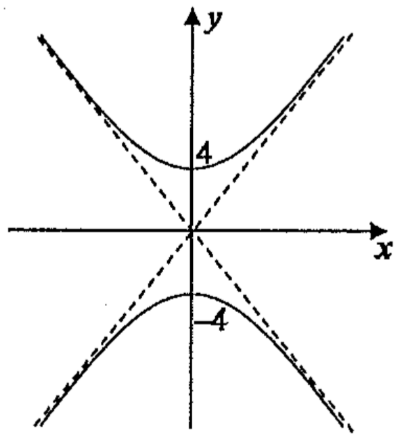
Section A - Multiple choice (10 marks)
(Mark the correct answer on the sheet provided.)

1. Which is a possible equation of the following hyperbola?



A) $9x^2 - 16y^2 = 144$

B) $16x^2 - 9y^2 = 144$

C) $9x^2 - 16y^2 = -144$

D) $16x^2 - 9y^2 = -144$

2. $\dfrac{2-i}{-2-i} = \ ?$

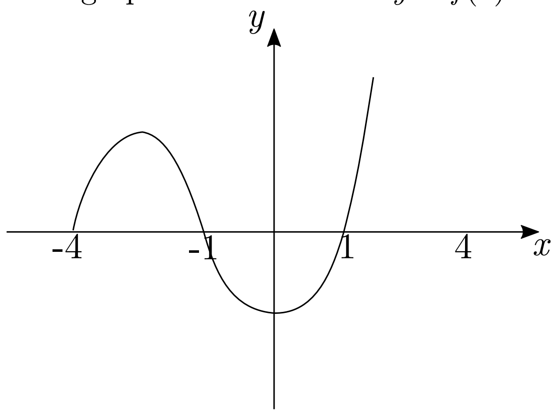
A) $-\dfrac{3}{5} + \dfrac{4}{5}i$

B) -1

C) $-1 + \dfrac{4}{3}i$

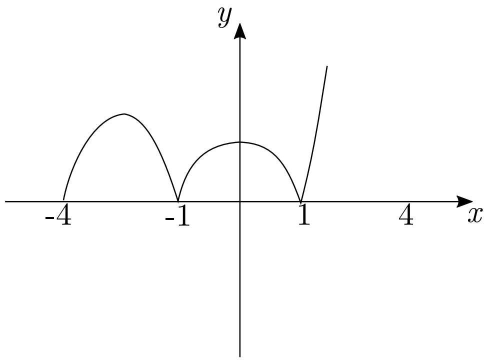
D) $-\dfrac{5}{3}$

3. The diagram shows the graph of the function $y = f(x)$.

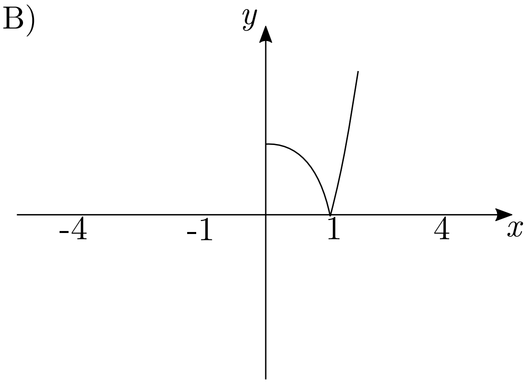


Which of the following is the graph of $y = f(|x|)$?

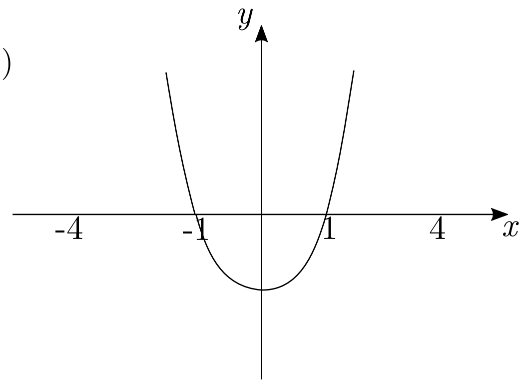
A)



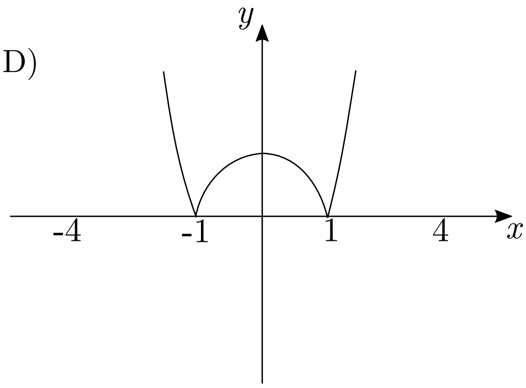
B)



C)



D)



4. The polynomial equation $x^3 + Ax^2 + B = 0$ has roots α, β and γ. What are the roots of the polynomial equation $(3x+2)^3 + A(3x+2)^2 + B = 0$?

A) $\dfrac{\alpha}{3} - 2, \dfrac{\beta}{3} - 2, \dfrac{\gamma}{3} - 2$

B) $\dfrac{\alpha - 2}{3}, \dfrac{\beta - 2}{3}, \dfrac{\gamma - 2}{3}$

C) $3\alpha + 2, 3\beta + 2, 3\gamma + 2$

D) $\alpha + \dfrac{2}{3}, \beta + \dfrac{2}{3}, \gamma + \dfrac{2}{3}$

5. Given the rectangular hyperbola $xy = 25$, which is the correct equation of its directrices?

A) $x + y = \pm\frac{5}{\sqrt{2}}$

B) $x + y = \pm 5$

C) $x + y = \pm 5\sqrt{2}$

D) $x + y = \pm 10$

6. Let α, β and γ be the zeroes of the polynomial $x^3 + 5x - 3$. Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

A) -125

B) 0

C) 9

D) 34

7. The equation $|z-3| + |z+3| = 10$ defines an ellipse. What is the length of the semi-minor axis?

A) 4

B) 5

C) 8

D) 10

8. The complex number z satisfies the equation $|z-2| = 1$. What is the maximum value of $\arg(z)$?

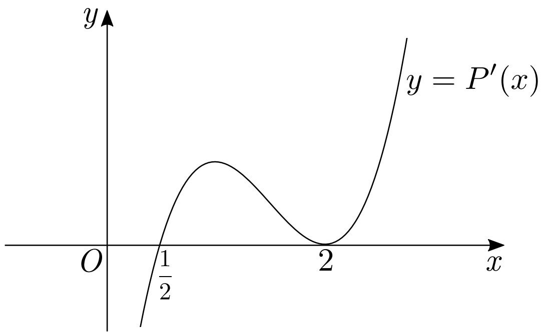
A) $\tan^{-1}\left(\frac{1}{2}\right)$

B) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

C) $\tan^{-1} 1$

D) $\tan^{-1}\sqrt{3}$

9. The following diagram shows the graph $y = P'(x)$, the derivative of a polynomial $P(x)$.



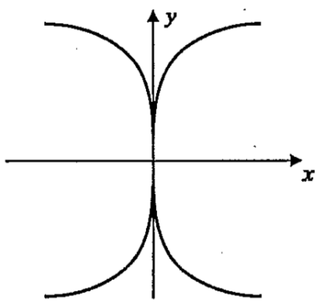
Which of the following expressions could be $y = P(x)$?

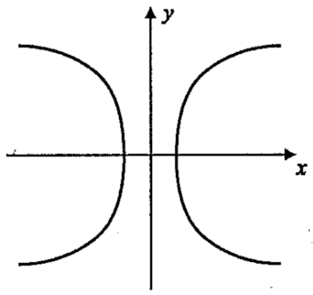
A) $x(x-2)^2$ B) $x(x-2)^3$

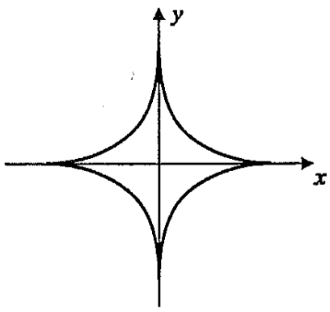
C) $(2x-1)(x-2)^2$ D) $(2x-1)(x-2)^3$

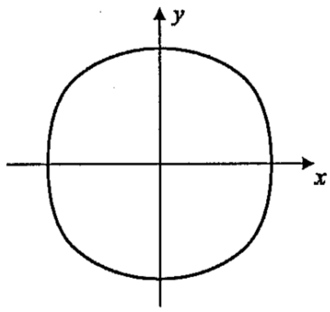
10. After differentiating a relation implicitly, we find that $\dfrac{dy}{dx} = \dfrac{-\sqrt[3]{y}}{\sqrt[3]{x}}$.

Which of the following could be a graph of this relation?

A)


B)


C)


D)


(End of Section A. Question 11 begins on the next page.)

Section B (35 marks)

Question 11 (Begin and label a new booklet.) **(15 marks)**

a) Evaluate $\int_1^e \log_e x \, dx$. [2]

b) Evaluate $\int_0^\pi \sin^3 x \, dx$. [3]

c) i) Find values A, B and C so that $\dfrac{x^2+x+1}{x^3+3x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3}$. [3]

ii) Hence find $\int \dfrac{x^2+x+1}{x^3+3x^2} \, dx$. [1]

d) Find $\int \dfrac{dx}{2+\sin 2x}$ using the substitution $t = \tan x$. Leave your answer in terms of t. [3]

e) The polynomial $P(x) = x^5 + 2x^4 + ax^3 + bx^2$ has $(x-1)^2$ as a factor.

Show that $a = -7$ and $b = 4$. [3]

(End of Question 11.)

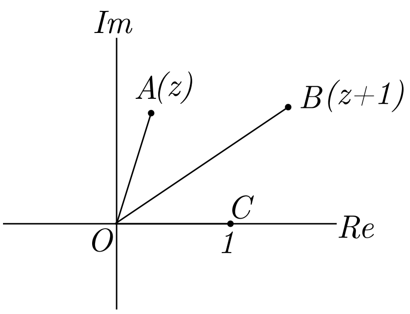
Question 12 (Begin and label a new booklet.) **(15 marks)**

a) Solve $z^2 = 5 - 12i$, giving your answer/s in the form $x + iy$. [2]

b) i) Express $-1 - i$ in modulus argument form. [2]

ii) Hence find the real part of $(-1-i)^{10}$. [2]

c) As shown, the points O, C, A and B on the Argand diagram represent the complex numbers $0, 1, z$ and $z+1$ respectively, where $z = \cos\theta + i\sin\theta$, $0 < \theta < \pi$. Copy the diagram.



i) Explain why $OCBA$ is a rhombus. [1]

ii) Draw the vector $z - 1$ on the diagram and hence explain why $\dfrac{z-1}{z+1}$ is purely imaginary. [2]

iii) Find, in terms of θ, the modulus and argument of $z+1$. [2]

d) Let ω be a non-real root of $z^7 - 1 = 0$.

i) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$. [1]

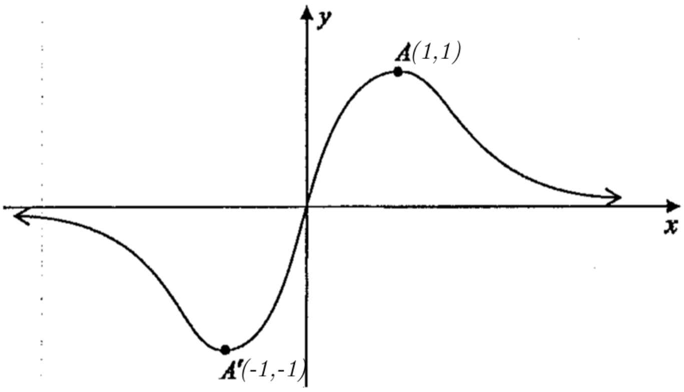
ii) Simplify $(\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3)$. [2]

iii) Sketch and label on the Argand diagram all seven roots of $z^7 - 1 = 0$. (You are not required to derive them.) [1]

(End of Question 12.)

Question 13 (Begin and label a new booklet.) **(15 marks)**

a) Drawn below is the graph of $y = \dfrac{2x}{1+x^2}$. Stationary points at A and A' are labeled as shown.



Sketch on separate axes, labeling any important features:

i) $y = \dfrac{|2x|}{1+x^2}$ [1]

ii) $y = \dfrac{1+x^2}{2x}$ [2]

iii) $y^2 = \dfrac{2x}{1+x^2}$ [2]

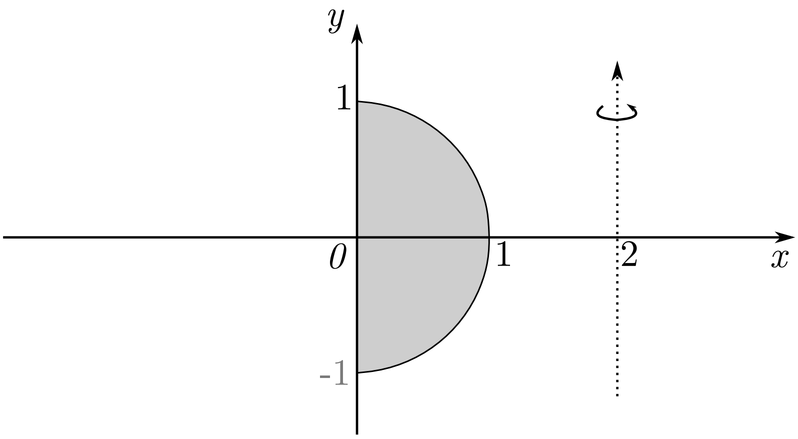
iv) $y = \log_e\left(\dfrac{2x}{1+x^2}\right)$ [2]

b) Sketch the region on the Argand diagram that satisfies:

$$-\frac{2\pi}{3} \le \arg(z-2) \le 0 \text{ and } \operatorname{Im}(z) \le -2\sqrt{3}$$ [2]

(Question 13 continues on the next page...)

c) The shaded semicircle in the diagram below is rotated about the line $x = 2$.



i) Using the method of cylindrical shells, show that the volume V of the resulting solid is given by:

$$V = \int_0^1 4\pi(2-x)\sqrt{1-x^2}\ dx\,.$$ [3]

ii) Hence find the volume of the solid. [3]

(End of Question 13.)

Question 14 (Begin and label a new booklet.) **(15 marks)**

a) Suppose that x and y are positive. Prove that $\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}$. [3]

b) An object of mass 5 kg is dropped in a medium where the resistance at speed v m/s has a magnitude of v Newtons. The acceleration due to gravity is 10 m/s^2.

i) Taking downwards as the positive direction, draw a force diagram and show that the equation of motion is $\ddot{x}=\frac{50-v}{5}$. [2]

ii) Find an expression for the velocity v at time t seconds after the object is dropped. [2]

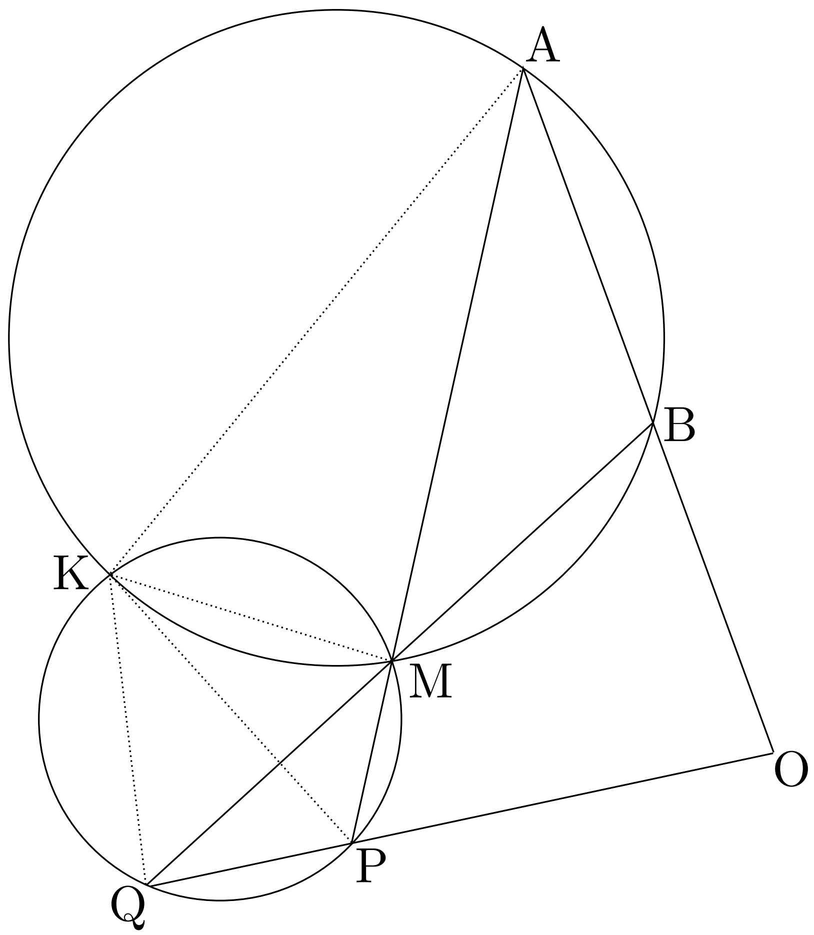
iii) Find the terminal velocity of the object. [1]

iv) Show that the distance x metered travelled when the speed is v m/s is given by:

$$x=250\log_e\left(\frac{50}{50-v}\right)-5v$$ [2]

(Question 14 continues on the next page...)

c) The diagram above shows two circles intersecting at K and M. From points A and B on the outer arc of one circle, lines are drawn through M to meet the other circle at P and Q respectively. The lines AB and QP meet at O.



i) Let $\theta = \angle KAB$, and give a reason why $\angle KMQ = \theta$.
[1]

ii) Prove $AKPO$ is a cyclic quadrilateral. [2]

iii) Let $\phi = \angle AKM$. Show that if $OBMP$ is a cyclic quadrilateral, then the points A, K and Q are collinear. [2]

(End of Question 14.)

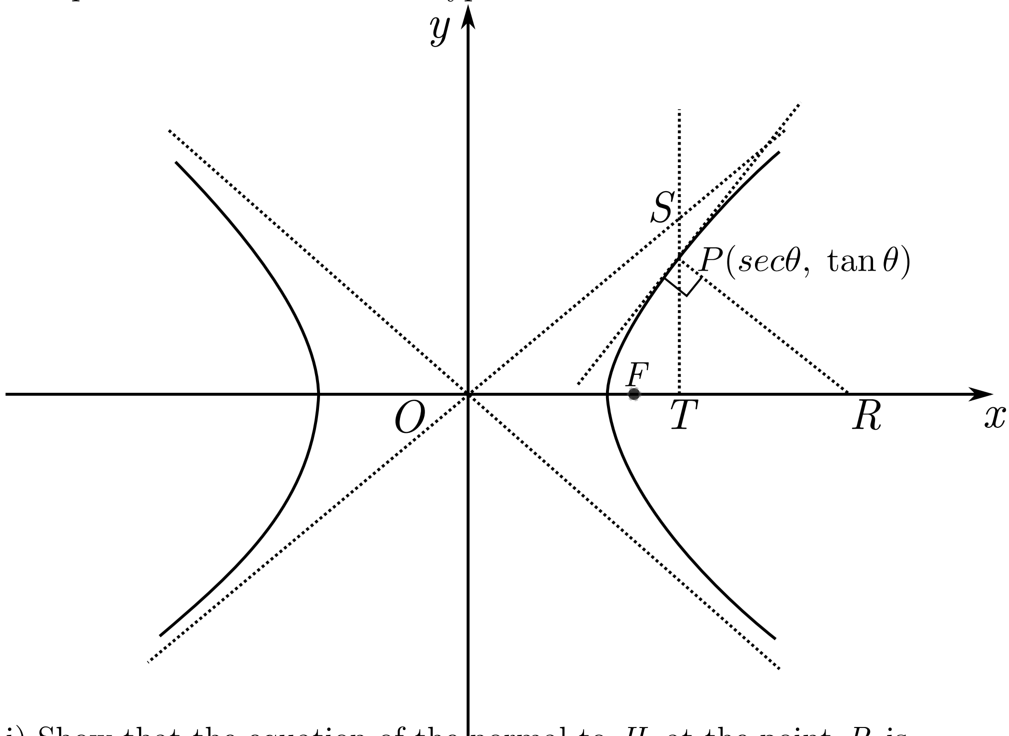
Question 15 (Begin and label a new booklet.) **(15 marks)**

a) Suppose that a, b and c are the side lengths of a triangle.

i) Explain why $(b-c)^2 < a^2$. [1]

ii) Deduce that $(a+b+c)^2 < 4(ab+bc+ca)$ [2]

b) The point $P(\sec\theta, \tan\theta)$ lie on the hyperbola with equation $x^2 - y^2 = 1$. A vertical line through P intersects with an asymptote at S and with the x-axis at T as shown. The normal to the hyperbola at P intersects the x-axis at R. The point F is a focus of the hyperbola.



i) Show that the equation of the normal to H at the point P is $y = -x\sin\theta + 2\tan\theta$. [2]

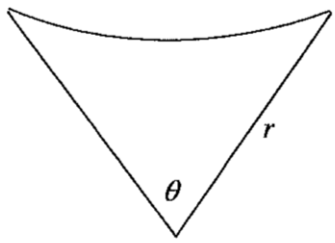
ii) Show that $RS = \sqrt{2} \times RT$. [2]

iii) Find the coordinates of the point W which lies on SR such that TW is parallel to the asymptote on which S lies. [2]

iv) For what values of θ will FW be the perpendicular bisector of SR? [1]

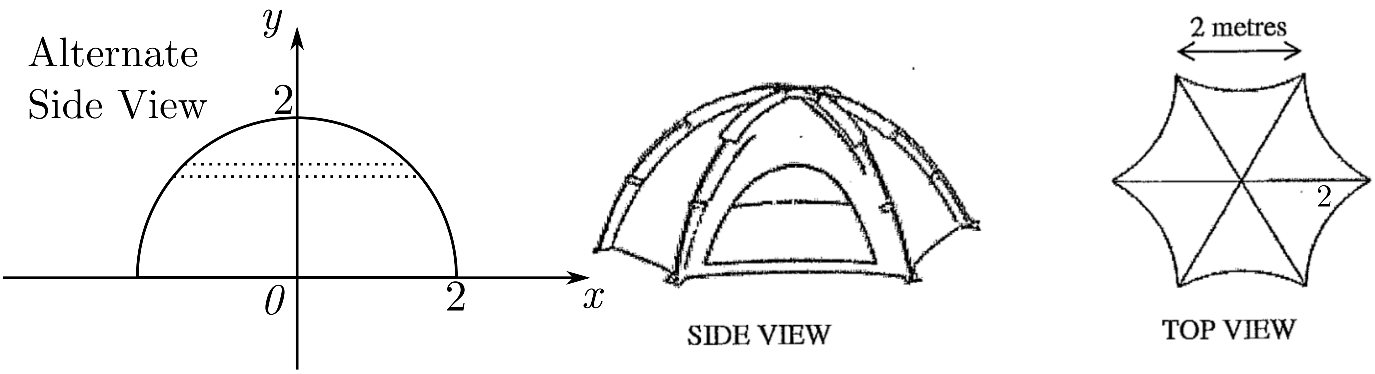
(Question 15 continues on the next page...)

c) i) A sector is normally made up of a triangle and a minor segment of a circle. The shape below is made by subtracting the area of the minor segment from the triangle.



Show that its area is given by $A = r^2\left(\sin\theta - \dfrac{\theta}{2}\right)$. [1]

ii) A dome tent is built with a base made up of six congruent copies of the shape from (i), each with radius 2 metres. The tent is supported by flexible exterior poles extended between opposite corners in semi-circular arcs.



iii) By taking slices parallel to the base of the tent, show that the volume enclosed by the tent is $\left(16\sqrt{3} - \dfrac{16\pi}{3}\right)$ cubic metres. [4]

(End of Question 15.)

Question 16 (Begin and label a new booklet.) **(15 marks)**

a) Consider the integral $I_n = \int_0^1 x^{2n+1} e^{-x^2}\, dx$.

i) Use integration by parts to show that $I_n = -\frac{1}{2e} + nI_{n-1}$, for $n \geq 1$. [3]

ii) Show that $I_0 = \frac{1}{2} - \frac{1}{2e}$. [1]

iii) Prove by mathematical induction that for all $n \geq 1$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} = e - \frac{2eI_n}{n!}.$$ [4]

iv) By considering the value of $x^{2n+1}e^{-x^2}$ in the domain $0 \leq x \leq 1$, explain why:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e$$ [1]

b) The numbers x, y and z satisfy:

$$x + y + z = 5$$
$$x^2 + y^2 + z^2 = 8$$
$$x^3 + y^3 + z^3 = 13$$

i) Show that $xy + xz + yz = \frac{17}{2}$. [1]

ii) Show that $x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 = 27$. [2]

iii) Hence show that $xyz = \frac{31}{6}$. [1]

iv) Use the previous parts to evaluate $x^4 + y^4 + z^4$. [2]

(End of Question 16.)

End of exam.

Extension 2 2019 Trial Solutions

Section A

Q1. $x=0,\ y=\pm4$

$\therefore 16x^2-9y^2=-144$

$\frac{y^2}{16}-\frac{x^2}{9}=1$ $\therefore$ (D)

Q2. $\frac{2-i}{-2-i}=\frac{(2-i)(-2+i)}{4+1}$

$=\frac{-4+2i+2i+1}{5}$

$=\frac{-3}{5}+\frac{4i}{5}$ (A)

Q3. C $y=f(|x|)$ (C)

Q4. $\alpha^3+A\alpha^2+B=0$

$\therefore$ If $x=\boxed{\frac{\alpha-2}{3}}$ $3x+2=\alpha$ ...

satisfies new equation (B)

Q5. $y-\frac{5}{\sqrt2}=-(x-\frac{5}{\sqrt2})$

$x+y=\frac{5}{\sqrt2}\times2$

$x+y=\sqrt2\times5$ (C)

Q6. $\alpha^3+5\alpha-3=0$

$\beta^3+5\beta-3=0$

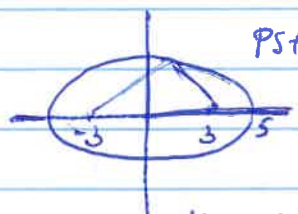
$\gamma^3+5\gamma-3=0$

$\alpha^3+\beta^3+\gamma^3+5(\alpha+\beta+\gamma)-9=0$

$\alpha+\beta+\gamma=0$

$\therefore \alpha^3+\beta^3+\gamma^3=9$ (C)

Q7.



$PS+PS'=2a$

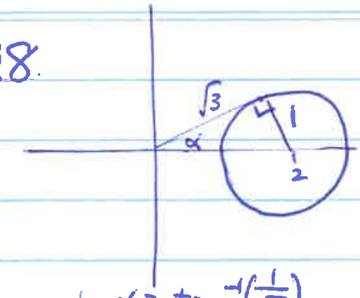
$a=5$ $b^2=a^2(1-e^2)$

$ae=3$ $=25(1-\frac{9}{25})$

$e=\frac{3}{5}$ $=16$

$b=4$ (A)

Q8.



$\therefore \alpha=\tan^{-1}(\frac{1}{\sqrt3})$ (B)

Q9. B or D since root of mult. 3 at $x=2$.

Try: $y=x(x-2)^3$

$y'=(x-2)^3+x\cdot3(x-2)^2$

$=(x-2)^2(x-2+3x)$

$=(x-2)^2(4x-2)$ ✓ (B)

(The other will not give suitable y')

Q10. at $x\geq0$, $y>0$,

$\frac{dy}{dx}<0$, $\therefore$ NOT A, B

C+D: at $x=0$, $y\neq0$, $\frac{dy}{dx}\to-\infty$ (in 1st quad) (C)

vertical at $x=a$

1	2	3	4	5	6	7	8	9	10
D	A	C	B	C	C	A	B	B	C

Section B

Q11 a) $\int_1^e \ln x \, dx$

$= [x\ln x]_1^e - \int_1^e x \cdot \frac{1}{x} \, dx$

$= (e\ln e - 1.\ln 1) - \int_1^e 1 \, dx$

$= e - [x]_1^e$

$= e - (e-1)$

$= 1$

b) $\int_0^{\pi} \sin^3 x \, dx$

$= -\int_0^{\pi} -\sin x(1-\cos^2 x) dx$ $\quad u = \cos x$

$du = -\sin x \, dx$

$= -\int_1^{-1} 1-u^2 \, du$ $\quad x=0, u=1$

$x=\pi, u=-1$

$= \int_{-1}^{1} 1-u^2 \, du$

$= [u - \frac{1}{3}u^3]_{-1}^{1}$

$= (1 - \frac{1}{3}(1)) - (-1 - \frac{1}{3}(-1))$

$= \frac{2}{3} - -\frac{2}{3}$

$= \frac{4}{3}$

c) i) $A(x+3)x + B(x+3) + Cx^2 = x^2+x+1$

$x=-3$: $9C = 9-3+1$

$9C = 7$

$C = \frac{7}{9}$ ✓

$x=0$: $3B = 1$

$B = \frac{1}{3}$ ✓

Coeff x^2:

$A + C = 1$

$A = 1 - C = \frac{2}{9}$ ✓

ii) $\int \frac{2}{9} \cdot \frac{1}{x} + \frac{1}{3} \cdot \frac{1}{x^2} + \frac{7}{9} \cdot \frac{1}{x+3} \, dx$

$= \frac{2}{9}\ln x - \frac{1}{3} \cdot \frac{1}{x} + \frac{7}{9}\ln(x+3) + C$

d) $\int \frac{dx}{2+\sin 2x}$ $\quad t = \tan x \rightarrow x = \tan^{-1} t$

$\sin 2x = \frac{2t}{1+t^2}$

$dx = \frac{dt}{1+t^2}$

$= \int \frac{1}{(2+\frac{2t}{1+t^2})} \cdot \frac{dt}{1+t^2}$

$= \int \frac{dt}{2+2t^2+2t}$

$= \frac{1}{2}\int \frac{dt}{t^2+t+1}$

$= \frac{1}{2}\int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$

$= \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$

$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$

e) $P(1) = 0$:

$1+2+a+b = 0$

$a+b = -3$ ... ①

$P'(1) = 0$

$P'(x) = 5x^4 + 8x^3 + 3ax^2 + 2bx$

$\therefore 5+8+3a+2b = 0$

$3a+2b = -13$ ... ②

② − 2×①: $a = -13 - -6$

$= -7$

$\therefore -7 + b = -3$

$b = 4$

Q12

a) $z^2 = 5 - 12i$

$(x+iy)^2 = 5 - 12i$, $x, y \in \mathbb{R}$

$x^2 - y^2 = 5$

$2xy = -12$

$\therefore y = \frac{-6}{x}$

$x^2 - \frac{36}{x^2} = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2 - 9)(x^2 + 4) = 0$

$\therefore x = \pm 3$

$x = 3,\ y = -2$

$x = -3,\ y = 2$

$\therefore z = 3 - 2i,\ -3 + 2i$

b) i)

$|z| = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$\arg z = \frac{-3\pi}{4}$

$\therefore -1 - i = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

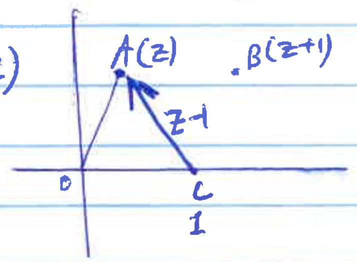
ii) $(\sqrt{2})^{10}\,\text{cis}\left(\frac{-30\pi}{4}\right)$ $\ \ +\frac{32\pi}{4}$

$= 32\,\text{cis}\left(\frac{2\pi}{4}\right)$

$= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\therefore$ real part $= 0$.

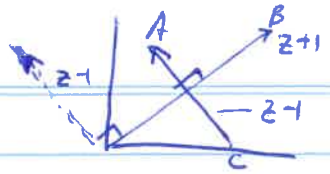
c)



i) $|z| = 1$, $\therefore$ all sides of parallelogram are equal to 1

$\therefore$ Rhombus.

c) ii) see diagram.

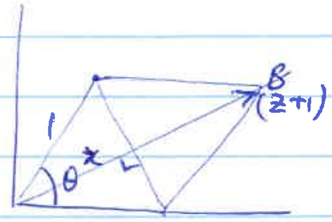


$\arg\left(\frac{z-1}{z+1}\right) = \arg(z-1) - \arg(z+1)$

$= 90^\circ \left(\frac{\pi}{2}\right)$ since diagonals of rhombus are $\perp$.

$\therefore \frac{z-1}{z+1} = ki$ $\ k \in \mathbb{R}$ ie purely imaginary.

iii)



$\arg(z+1) = \frac{\theta}{2}$ (diagonals of rhombus bisect $\angle$ it passes)

$\frac{x}{1} = \cos\frac{\theta}{2}$

$\therefore |z+1| = 2\cos\frac{\theta}{2}$.

d) i) $z^7 - 1 = 0$

$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

Since ω is non real, $\omega \neq 1$

$\therefore \omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$.

ii) $\omega^7 + \omega^6 + \omega^4 + \omega^8 + \omega^7 + \omega^5 + \omega^{10} + \omega^9 + \omega^7$

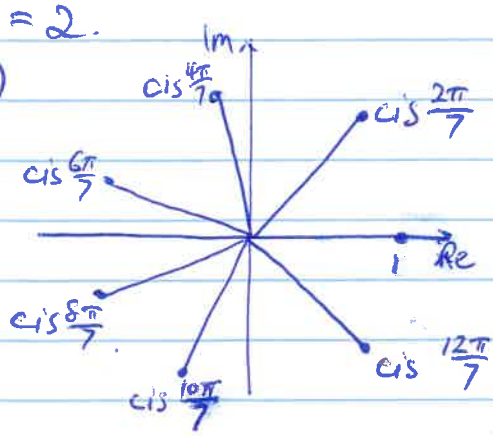
$= 1 + \omega^6 + \omega^4 + \omega + 1 + \omega^5 + \omega^3 + \omega^2 + 1$

$= 3 + (\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega)$ $\quad$ since $\omega^7 = 1$

$= 3 - 1$

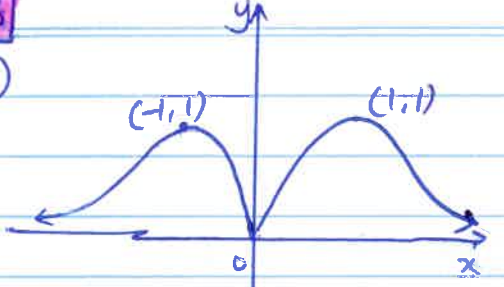
$= 2$.

iii)

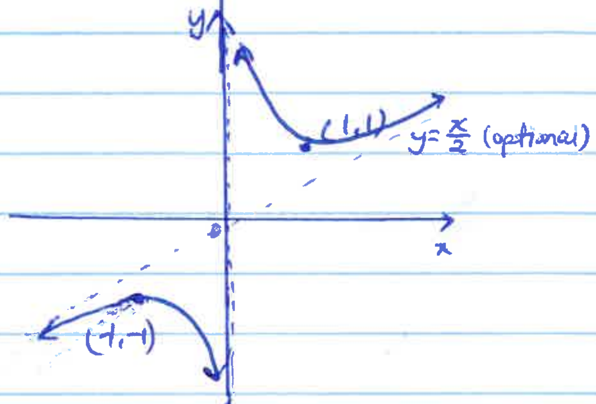


Q13

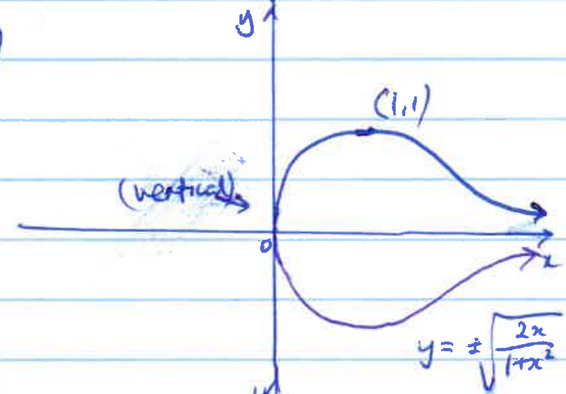
a) i)



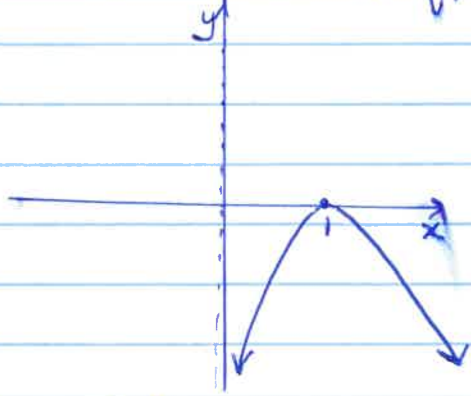
ii)



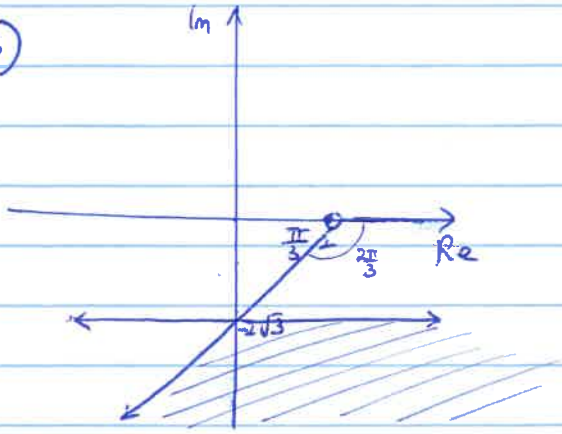
iii)



iv)



b)



c) i) $\delta V = 2\pi r h \,\delta x$

$r = 2 - x$

$h = 2y$

$= 2\sqrt{1-x^2}$ from $x^2 + y^2 = 1$

$\delta V = 2\pi(2-x) \cdot 2\sqrt{1-x^2}\,\delta x$

$= 4\pi(2-x)\sqrt{1-x^2}\,\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \delta V$

$= 4\pi \int_0^1 (2-x)\sqrt{1-x^2}\,dx$ as req.

ii) $V = 8\pi \int_0^1 \sqrt{1-x^2}\,dx - 4\pi \int_0^1 x\sqrt{1-x^2}\,dx$

$= 8\pi \cdot \frac{\pi \times 1^2}{4} + 2\pi \int_0^1 -2x\sqrt{1-x^2}\,dx$

$u = 1 - x^2$

$du = -2x\,dx$

$x = 0,\ u = 1$

$x = 1,\ u = 0$

$= 2\pi^2 + 2\pi \int_1^0 \sqrt{u}\,du$

$= 2\pi^2 + 2\pi \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^0$

$= 2\pi^2 + 2\pi\left(0 - \frac{2}{3}\right)$

$= 2\pi^2 - \frac{4\pi}{3}$ units^3

Q14

a) RTP $\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}$

~~$\frac{y+x}{xy}\geq\frac{4}{x+y}$~~

~~$(x+y)^2\geq 4xy$~~

~~$x^2+2xy+y^2\geq 4xy$~~ ✓

$(x-y)^2\geq 0$

$x^2-2xy+y^2\geq 0$

$+4xy \quad +4xy$

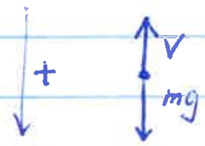
$x^2+2xy+y^2\geq 4xy$

$(x+y)^2\geq 4xy$

$\frac{x+y}{xy}\geq\frac{4}{x+y}$ (since $x,y>0$)

$\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}$ as req.

b) i)



$F_{net}=mg-v$

$ma=mg-v$

$5a=50-v$

$a=\frac{50-v}{5}$

ii) $\frac{dv}{dt}=\frac{50-v}{5}$

$\frac{dt}{dv}=\frac{5}{50-v}$

$t=-5\int\frac{-1}{50-v}dv$

$t=-5\ln(50-v)+c$

$t=0,\ v=0$

$0=-5\ln 50+c$

$c=5\ln 50$

$t=-5\ln(50-v)+5\ln 50$

$t=-5\ln\left(\frac{50-v}{50}\right)$

$\frac{50-v}{50}=e^{-\frac{t}{5}}$

$50-v=50e^{-\frac{t}{5}}$

$v=50-50e^{-\frac{t}{5}}$

b) iii) $a\to 0,\ 0=\frac{50-v}{5}$

$\therefore$ $v\to 50$ m/s is terminal velocity.

iv) $v\frac{dv}{dx}=\frac{50-v}{5}$

$\frac{dv}{dx}=\frac{50-v}{5v}$

$\frac{dx}{dv}=\frac{5v}{50-v}$

$x=\int\frac{5v}{50-v}dv$

$=-5\int\frac{-v+50-50}{50-v}dv$

$=-5\int 1\,dv-250\int\frac{-1}{50-v}dv$

$x=-5v-250\ln(50-v)+c$

$x=0,\ v=0$: $c=250\ln 50$

$x=-5v-250\ln(50-v)+250\ln 50$

$=-5v+250\left(\frac{50}{50-v}\right)$ as req.

c) i) $\angle KMQ=\angle KAB=\theta$ (exterior $\angle$ of cyclic quad is equal to interior ~~possib~~ opposite $\angle$)

ii) $\angle KPQ=\angle KMQ=\theta$ ($\angle$s standing on same arc are equal)

$\therefore$ $\angle KPO=180-\theta$ ($\angle$s on a straight line)

$\therefore$ AKPO is cyclic (opposite $\angle$s add to 180°)

iii) $\angle ABM=180-\phi$ (opposite $\angle$s of cyclic quad add to 180°)

$\angle MPO=180-\phi$ (exterior $\angle$ of cyclic quad is equal to interior opposite $\angle$) (BMPO)

$\therefore$ $\angle QKM=180-\phi$ (- - -) (KMPQ)

$\therefore$ $\angle QKA=180°$ (from $\angle AKM+\angle QKM$)

$\therefore$ AKQ is collinear if BMPO is cyclic.

Q15

a) i) The triangle inequality:

$a+b>c \Rightarrow a>c-b$

(only if $c\geq b$:) $a^2>(c-b)^2$

$a+c>b \Rightarrow a>b-c$

(only if $b\geq c$:) $a^2>(b-c)^2$

In either case $a^2>(b-c)^2$

(Arguments using $|b-c|$ accepted)
(if diagram or cases shown)

ii) $a^2>(b-c)^2$

$a^2>b^2-2bc+c^2$

Similarly $b^2>a^2-2ac+c^2$

$c^2>b^2-2ab+a^2$ (+)

$a^2+b^2+c^2>2a^2+2b^2+2c^2-2(ab+ac+bc)$

$2ab+2ac+2bc>a^2+b^2+c^2$

$(+2ab+2ac+2bc)$ $(+2ab+2ac+2bc)$

$4(ab+bc+ca)>(a+b+c)^2$ as req.

b) i) $x^2-y^2=1$, $a=1$, $b=1$

$b^2=a^2(e^2-1)$

$e^2=2$

$e=\sqrt{2}$

$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$ $P(\sec\theta,\tan\theta)$

$=\frac{\sec^2\theta}{\sec\theta\tan\theta}$

$=\frac{\sec\theta}{\tan\theta}$

$=\frac{1}{\sin\theta}$

$m_\perp=-\sin\theta$

eqn normal: $y-\tan\theta=-\sin\theta(x-\sec\theta)$

$y-\tan\theta=-x\sin\theta+\tan\theta$

$y=-x\sin\theta+2\tan\theta$

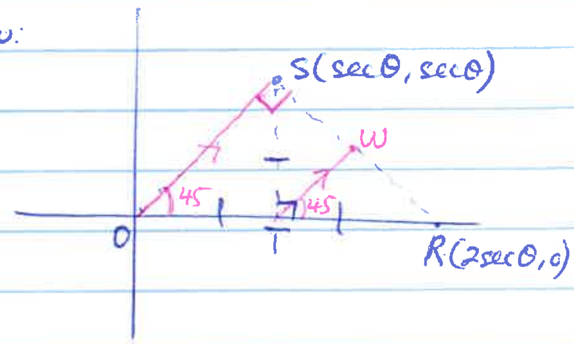
as req.

b) ii) Eqn OS: $y=x$

$\therefore$ at S, x-coord is $\sec\theta$

$\therefore S:(\sec\theta,\sec\theta)$

Now:



R: $y=0$: $0=-x\sin\theta+2\tan\theta$

$x\sin\theta=2\tan\theta$

$x=\frac{2}{\cos\theta}=2\sec\theta$

$\therefore$ On diagram above $\triangle STR$ is right isosceles: $\therefore \angle SRT=45^\circ$

$\therefore \frac{TR}{SR}=\sin 45^\circ$

$\therefore \sqrt{2}\cdot TR=RS$

iii) On diagram: $OT=TR \Rightarrow SW=WR$

(intercepts on transversals cut by parallel lines in same ratio)

$\therefore$ W is M_{SR}: $\left(\frac{\sec\theta+2\sec\theta}{2},\frac{\sec\theta}{2}\right)$

$W:\left(\frac{3}{2}\sec\theta,\frac{1}{2}\sec\theta\right)$

iv) TW is already perp. bisector of SR.

$\therefore$ F must be T:

$\therefore ae=\sec\theta$

$\sqrt{2}=\sec\theta$

$\theta=45^\circ$ or 315°

ie

$\frac{\pi}{4}$ or $\frac{7\pi}{4}$

Q15c)

c) i) Area segment $:= \frac{1}{2}r^2(\theta - \sin\theta)$

Area triangle $= \frac{1}{2}r^2\sin\theta$

$A = \frac{1}{2}r^2\sin\theta - \frac{1}{2}r^2(\theta - \sin\theta)$

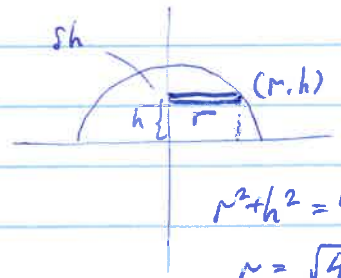
$= r^2\sin\theta - \frac{1}{2}r^2\theta$

$= r^2(\sin\theta - \frac{\theta}{2})$

ii) $\theta = \frac{\pi}{3}$

$\therefore A = r^2(\sin\frac{\pi}{3} - \frac{\pi}{6}) \times 6$

$= r^2(\frac{\sqrt{3}}{2} - \frac{\pi}{6}) \times 6$



$r^2 + h^2 = 4$

$r = \sqrt{4-h^2}$

$\therefore A = (\sqrt{4-h^2})^2(\frac{\sqrt{3}}{2} - \frac{\pi}{6}) \times 6$

$= (\frac{\sqrt{3}}{2} - \frac{\pi}{6})(4-h^2) \times 6$

$V = \lim_{\delta h \to 0} \sum_{h=0}^{2} A\,\delta h$

$= 6\int_0^2 (\frac{\sqrt{3}}{2} - \frac{\pi}{6})(4-h^2)\,dh$

$= 6(\frac{\sqrt{3}}{2} - \frac{\pi}{6})\left[4h - \frac{h^3}{3}\right]_0^2$

$= 6(\frac{\sqrt{3}}{2} - \frac{\pi}{6})(8 - \frac{8}{3})$

$= (\frac{\sqrt{3}}{2} - \frac{\pi}{6})(32)$

$= 16\sqrt{3} - \frac{16\pi}{3}$ units^3.

Q16

a) i) $I_n = \int_0^1 x^{2n}\, xe^{-x^2}\,dx$

$u = x^{2n}$ $\quad \therefore v = -\frac{1}{2}e^{-x^2}$

$u' = 2nx^{2n-1}$ $\quad v' = xe^{-x^2}$

$v = -\frac{1}{2}\int -2xe^{-x^2}dx$

$= -\frac{1}{2}e^{-x^2}$

$I_n = \left[x^{2n} \times -\frac{1}{2}e^{-x^2}\right]_0^1 - \int_0^1 -\frac{1}{2}e^{-x^2} \cdot 2n\,x^{2n-1}\,dx$

$= \frac{-1}{2e} + n\int_0^1 e^{-x^2} \cdot x^{2n-1}\,dx$

$= \frac{-1}{2e} + n\int_0^1 e^{-x^2} x^{2(n-1)+1}\,dx$

$= \frac{-1}{2e} + nI_{n-1}$ ✓ as req.

ii) $I_0 = \int_0^1 xe^{-x^2}\,dx$

$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1$

$= -\frac{1}{2}e^{-1} - (-\frac{1}{2}e^0)$

$= \frac{-1}{2e} + \frac{1}{2}$ as req.

iii) Let $n=1$

LHS $= 1 + \frac{1}{1!}$

$= 2$

RHS $= e - \frac{2eI_1}{1!}$

$= e - 2eI_1$

$I_1 = \frac{-1}{2e} + 1I_0$

$= \frac{-1}{2e} + \frac{-1}{2e} + \frac{1}{2}$

$= \frac{1}{2} - \frac{1}{e}$

RHS $= e - 2e(\frac{1}{2} - \frac{1}{e})$

$= e - e + 2$

$= 2.$

16a) iii) cont

Assume for $n=k$:

$1+\frac{1}{1!}+\ldots+\frac{1}{k!}=e-\frac{2eI_k}{k!}$

RTP for $n=k+1$

$1+\frac{1}{1!}+\ldots+\frac{1}{k!}+\frac{1}{(k+1)!}=e-\frac{2eI_{k+1}}{(k+1)!}$

$LHS=e-\frac{2eI_k}{k!}+\frac{1}{(k+1)!}$

$=e-\frac{2eI_k(k+1)-1}{(k+1)!}$

Now from i): $I_{k+1}=-\frac{1}{2e}+(k+1)I_k$

$\therefore 2eI_{k+1}=-1+2e(k+1)I_k$

$=e-\frac{2eI_{k+1}}{(k+1)!}$ as required.

true for $n\geq 1$ by mathematical induction.

iv) For $0\leq x\leq 1$, $0\leq x^{2n+1}\leq 1$

and $0\leq e^{-x^2}\leq 1$

$I_n=\int_0^1 x^{2n+1}e^{-x^2}dx\leq\int_0^1 1\,dx$ (upper bound)

$0\leq I_n\leq 1$ for $n\geq 0$

$\therefore$ As $n\to\infty$

$\frac{2eI_n}{n!}$ vanishes.

$\therefore e-\frac{2eI_n}{n!}\to e$ as $n\to\infty$

16b) i) $xy+xz+zy=\frac{(x+y+z)^2-(x^2+y^2+z^2)}{2}$

$=\frac{5^2-8}{2}$

$=\frac{17}{2}$.

ii) consider: ~~$(xy+xz+zy)(x+y+z)$~~

$(x+y+z)(x^2+y^2+z^2)=5\times 8$

$x^3+xy^2+xz^2+y^3+x^2y+z^2y+z^3+x^2z+y^2z=40$

$\therefore xy^2+xz^2+yz^2+yx^2+zx^2+zy^2=40-(x^3+y^3+z^3)$

$=40-13$

$=27$

Q16b cont)

iii) Consider:

$(x+y+z)(xy+xz+yz)$

$= xy^2 + x^2z + xyz + xy^2 + xyz + y^2z + xyz + xz^2 + yz^2$

$= 27 + 3xyz$ (ii)

$\therefore 5\left(\frac{17}{2}\right) = 27 + 3xyz$

↑ (i)

$\therefore \frac{85}{2} - 27 = 3xyz$

$\frac{31}{2} = 3xyz$

$xyz = \frac{31}{6}$

iv) To find $x^4+y^4+z^4$ first consider:

$(x+y+z)(x^3+y^3+z^3) = \underbrace{x^4+y^4+z^4}_{X} + \underbrace{xy^3+xz^3+yx^3+yz^3+zx^3+zy^3}_{① \; Y}$

$5 \times 13 = X + Y$

To find ① consider:

$(xy+xz+yz)(x^2+y^2+z^2) = x^3y + xy^3 + xyz^2 + x^3z + xzy^2 + xz^3 + yzx^2 + y^3z + yz^3$

$\frac{17}{2} \times 8$

$= \underbrace{x^3y + xy^3 + x^3z + xz^3 + y^3z + yz^3}_{① \; Y} + xyz(z+x+y)$

$\frac{31}{6} \times 5$

$\therefore \frac{17}{2} \times 8 = Y + \frac{31}{6} \times 5$

$Y = 68 - \frac{155}{6}$

$= \frac{253}{6}$

$\therefore X = 65 - \frac{253}{6}$

$= \frac{137}{6}$